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A Geometrical Treatment of Singular Trajectories

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The nonlinear field equations often arising in geometrodynamical theories of matter generally exhibit nonremovable singularities. Assuming that the field equations are either (1) analytic, or (2) structurally stable, we show that the Christoffel symbols of the second kind have certain properties (4), (9). The singularities are such that wedge-shaped sets can be found containing n-parameter families of trajectories emanating from a given point on a singularity. In particular cases where the singularity is an isolated point, entire neighborhoods have been found, composed of trajectories. The latter situation is especially convenient in that a generalized tangent space can be defined, in which various manipulations of other field equations can be done (separation of variables, potential theory) and for which an exponential map can be set up. We show that under (4) the geodesics (trajectories) vary continuously with respect to limit tangent vectors at the singularity. Under a slightly stronger condition (23), trajectories vary differentiably with respect to limit tangent vectors. The limit tangent vectors are the elements of the generalized tangent space.

INTRODUCTION

Coupled systems of nonlinear differential equations displaying a fair degree of complexity at singularities arise not infrequently in particle physics [1] and biophysics [2, 3] to mention only a few applications. In our work, a geometrical approach has been taken with singular trajectories. Some results of the program will be presented which center around certain conditions, (13), (23), under which trajectories are uniquely and continuously (respectively, differentiably) determined by initial conditions at the singularities. These initial conditions will be shown to generalize the notion of a tangent space so that a kind of exponential map can be defined, which enables much analysis to be done.

We first introduce a concept of "associated dynamical system," something like geodesic equations for a metric in which the trajectories are geodesics. As a preliminary result to motivate these notions, we show that structurally stable systems in up to four controls satisfy the hypotheses of main Theorems 1 and 2. The resulting "limit tangent vectors" which compose the

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generalized tangent space are then further motivated by an application to solutions of

$$dx/dt = f(x),$$

where f is analytic and has no roots of order 2. As a further application, we prove a generalized Morse Index Theorem for a special form of the associated dynamical system for which trajectories are a Jacobi variation of geodesics on a certain (possibly incomplete) Riemannian manifold. Under the hypotheses of Theorem 1 there can exist at most finitely many conjugate points along a maximal geodesic of finite length. Finally, it is shown that conditions (13) and (23) hold for geodesics on algebraic varieties in R^n.

Definitions and Preliminary Results

A *dynamical system* is considered with notation $x_1, \ldots, x_n$ for the state variables in domain X, and $a_1, \ldots, a_m$ control variables in domain A. The system equations are thus

$$\begin{aligned} dy_i/dt &= f_i(y_1, \ldots, y_n, a_1, \ldots, a_m, t), \qquad i = 1, \ldots, n, \\ z_j &= g_j(y_1, \ldots, y_n, a_1, \ldots, a_m, t), \qquad j = 1, \ldots, M, \end{aligned} \tag{1}$$

with unique solution guaranteed by appropriate conditions in some interval of time $[t_0, t_1]$ for given initial values $y_1(t_0), \ldots, y_n(t_0)$, $a_1(t_0), \ldots, a_m(t)$ in their respective domains A, X. The z_j may serve as measured quantities or as variables constrained to a domain, Z.

Suppose $f_i(0, a, 0) = 0$ for all $a \in A$ and i, and that there exists an N-parameter family of trajectories emanating from 0 varying continuously or with some degree of differentiability with respect to the parameters. It is convenient to have conditions under which these trajectories have tangent vectors in the limit at 0 and such that they vary with some degree of differentiability with respect to the limit tangent vectors. Such liberty to specify both initial point and tangent vector is a property of a second order ordinary differential equation, suggesting the following artifice:

Construction of Geodesic Associated System

Given an n'-parameter family of trajectories $(n' \leqslant n)$ of (1) varying differentiably with respect to parameters $p_1, \ldots, p_{n'}$ in some open domain U in $R^{n'}$ and all approaching a common point q in R^n, let T be the field of tangent vectors to the trajectories and X_i the variation vector field of the trajectories under variation of p_i. Clearly a metric g can be found such that the trajectories satisfy

$$\frac{dT_k}{dt} = \Gamma^k_{ij} T_i T_j, \qquad k = 1,..., n, \tag{2}$$

$$\frac{dX_{mk}}{dt} = \Gamma^k_{ij} T_i X_{mj}, \qquad m = 1,..., n', \tag{3}$$

where

$$\Gamma^k_{ij} \equiv g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m})$$

are the Christoffel symbols of the Riemannian connection. Any set of equations so obtained will be called a *geodesic associated system.* Equation (2) expresses that the geodesics are trajectories, and (3) that the variation vectors are parallel along the trajectories with respect to the connection.

Note that q is a common intersection point of the n'-parameter family. Suppose t is the arc length from q. Let q be a singular trajectory, and suppose

$$g(X_m(\gamma(t), X_m(\gamma(t)))) \to t^a \qquad \text{as} \quad t \to 0$$

for some real a and for trajectories γ parametrized in a sufficiently small compact neighborhood of any given set of values in U. Reparametrizing arc length t by s such that for some $b \in (0, 1)$, $t = s^{1-b}$, the tangent vector T becomes $(0,..., 0, s^{-b})$ in coordinates $(p_1,..., p_{n'}, s)$. It then follows from (3) that $s\Gamma^{n'}_{n'n'} = -bs^b$, which is integrable on $[0, \varepsilon]$ for any small ε. Furthermore, $dx_m/dt \to \frac{1}{2}at^{(1/2)a-1}$, where x_m is the mth and only nonzero component of X_m, since $x_m \to t^{(1/2)a}$ as $t \to 0$. Thus from (3),

$$s^b \tfrac{1}{2} a t^{(1/2)a-1} \to \Gamma^m_{n'm} s^{-b} x_m \to \Gamma^m_{n'm} s^{-b+(1/2)a(1-b)}$$

or

$$s\Gamma^m_{n'm} \to \tfrac{1}{2} a s^{2b-1},$$

which is also integrable on $[0, \varepsilon]$.

There are a number of advantages of an associated system. It will be shown possible to parametrize certain stable or unstable manifolds of trajectories by tangents at their critical points. With the geodesic associated system, it is further possible to define an exponential map when the trajectories can be parametrized by an m-sphere. The latter system will be useful here in the context of a Morse Index Theorem generalized to handle points where Christoffel symbols have infinite but integrable singularities.

To motivate the rather technical Theorem 2 concerning associated systems, we will first show that its main hypothesis is satisfied for the important class of structurally stable gradient systems (topology of trajectories invariant under diffeomorphisms).

Remarks. The introduction of the gradient is suitable in the context of associated systems in view of the density of the structurally stable gradient systems in the class of gradient systems. If an associated geodesic system can be chosen to have negative curvature in some neighborhood of a singular solution, then there exists a local potential [4].

Notation. Let $\max_s |f|$ be the maximum absolute value of the continuous function f over the wedge-shaped set $\{(x_1,..., x_n) \in R^n: s \leqslant x_n \leqslant \varepsilon, \sum_{i=1}^{n-1} x_i^2 \leqslant \lambda x_n^2\}$ for some fixed ε and λ. This wedge will be denoted W_s throughout.

THEOREM 1. *If* (1) *is structurally stable, with a critical manifold M of the potential F of f, and $m \leqslant 5$, then there exists an associated system at any point $p \in M$ with the property*

$$\int_0^\varepsilon \max_s |f| < \infty, \tag{4}$$

for some $\varepsilon > 0$, for any wedge-shaped set W_s.

Proof (*outline*). Transform the potential, F, to one of Thom's diffeomorphism class representatives [5]. The right-hand sides are polynomials without roots of order 2. The following proposition then guarantees integrability in W_s to obtain (4). The inverse-diffeomorphic F will clearly also satisfy (4).

PROPOSITION. *If the analytic function f has a root of order λ at* 0 *then there are coordinates and a wedge-shaped set lying off the zero set of f in which solutions $x(t)$ of* (1) *have the property*

$$|x(t-q)| \leqslant C(t)\, t^{-1/(1-\lambda)}, \tag{5}$$

for some C, $C(0) \neq 0$, and with $x(0) = q^{\lambda-1}$. When $\lambda = 1$, then dx/dt approaches a nonzero constant in these coordinates as t approaches 0.

Proof. It is only necessary to observe that a solution must be an analytic function, and to apply a theorem of Łojasiewicz [6] to the distance function $d(\bar{x}(t), S)$ of the trajectory from the zero set S.

Remarks. If $\lambda \neq 2$, solutions are integrable over bounded neighborhoods. Clearly the proposition would extend to f continuous, if "root" were in the following sense: $f(x) = x^\lambda g(x)$ for some continuous g such that $g(0) \neq 0$.

DEFINITION. A *limit tangent vector*, when it exists, for a trajectory approaching or leaving a critical point will be such that there exists an α for which

$$\lim_{t\to 0} t^{\alpha} \frac{dx_i}{dt} \neq 0 \tag{6}$$

for at least one component of the tangent vector to the trajectory, with respect to some coordinate system and parametrization of the trajectory. The remaining limits of components must exist.

Note. The above proposition assures that analytic systems have limit tangent vectors.

The initial motivation for introducing a corrective factor t^{α} to the tangent vectors was to perturb the solution field of (1) as little as possible around a singularity, say, at q. A natural choice for a global correction which becomes 1 at infinity is

$$z(t) \equiv \frac{1 + d(x, q)}{d(x, q)},$$

where d is a given metric. With $w(t) \equiv z(t)\, x(t)$,

$$\frac{dw}{dt} = zf\left(\frac{w}{z}, t\right) + f\left(\frac{w}{z}\right) \cdot \frac{w}{z} \Big/ \left|\frac{w}{z}\right|^3 \tag{7}$$

provides a nonsingular vector field which approaches a nonzero vector as t approaches 0. The goodness of the approximation can be improved by choosing in place of z some analytic function which is asymptotic to 1 at infinity and is 0 at q. If f also is analytic, then (7) may be expanded to

$$\frac{dw}{dt} = f_1(t) w + f_2(t) \frac{w^2}{z} + f_3(t) \frac{w^3}{z} + \cdots \tag{8}$$

with nonsingular terms. (We are indebted to C. Coleman for this idea.) Since dx/dt is the root of an analytic equation, it has asymptotic dependence t^{α} near 0.

In addition to use in the proofs, the following lemmas will show how the hypothesis of Theorem 2 can be manipulated.

LEMMA 1.

$$\int_0^{\epsilon} s \max_s \left|\frac{\partial g}{\partial x_i}\right| ds < \infty, \tag{9}$$

if and only if

$$\int_0^\epsilon \max_s |g|\, ds < \infty. \tag{10}$$

Proof. First note that for a differentiable function g, $(d/ds)(\max_s |g|)$ exists almost everywhere, and is either negative or 0. In either case,

$$\frac{d}{ds}(\max_s |g|) \geqslant -\max_s (\|\nabla g\|) \geqslant -n \max_s \left(\max_i \left|\frac{\partial g}{\partial x_i}\right|\right).$$

Using this inequality and integrating by parts,

$$\begin{aligned} -n\int_0^\epsilon s \max_s \left(\max_i \left|\frac{\partial g}{\partial x_i}\right|\right) ds &\leqslant \int_0^\epsilon s\frac{d}{ds}(\max_s |g|)\, ds \\ &= \varepsilon \max_\epsilon |g| - \lim_{s\to 0} (s \max_s |g|) - \int_0^\epsilon \max_s |g|\, ds. \end{aligned} \tag{11}$$

By hypothesis the left-hand side of (11) is greater than $-\infty$, and since $\max_s |g|$ is positive, $\lim_{s\to 0}\max_s |g|$ is positive, so $\int_0^\epsilon \max_s |g|\, ds$ is positive. But then, by inequality (11), the latter cannot be infinite. The converse is more trivial.

LEMMA 2. *Under the same conditions as Lemma* 1, *we have*

$$\int_0^\epsilon s(\max_s |g|)^2\, ds < \infty. \tag{12}$$

Proof. By two integrations by parts,

$$\begin{aligned} \int_0^\epsilon sf^2\, ds &= \left[sf\int f\, ds\right]_0^\epsilon - \int_0^\epsilon \left(\left(f + s\frac{df}{ds}\right)\int f\, ds\right) ds \\ &= sf\left(\int f\, ds\right) - \int \left(f\int f\, ds\right) ds - \int \left(s\frac{df}{ds}\left(\int f\, ds\right)\right) ds. \end{aligned} \tag{13}$$

All terms on the right are bounded by Lemma 1.

Notation. For some fixed ε and λ, let $W' \equiv \{(T_1,..., T_n) \in R^n: \sum_{i=1}^{n-1} T_i^2 \leqslant \bar{\lambda}^2 T_n^2,\ |T_i| \leqslant \bar{\lambda}\delta/n$ and $(1 + \lambda - \bar{\lambda})\delta \leqslant T_n \leqslant (2 - \lambda + \bar{\lambda})\delta\}$ for some δ and $\bar{\lambda}$ such that $0 < \delta \leqslant 1$ and $0 < \bar{\lambda} < \lambda$. Note that $W' \subseteq W_s$.

THEOREM 2. *Let $F_i: R^n - 0 \to R$ be continuously differentiable functions for $i = 1,..., N$ such that $\int_0^\epsilon s^\beta \max_s |\partial F_i/\partial x_j|\, ds < \infty$ for $\beta = 1 - \alpha$, and G_k:*

$R^{n-N} \to R$ be continuous functionals, uniformly continuous in t which satisfy the Lipshitz condition. Then the system of differential equations

$$\frac{d^2x_k}{dt^2} = G_k\left[F_1(x),\ldots, F_N(x), \frac{dx_1}{dt},\ldots, \frac{dx_n}{dt}, t\right], \tag{14}$$

for $k = 1,\ldots, n$, has a unique solution for $t \in [0, \varepsilon']$ for some ε', specified by the initial conditions

$$\lim_{t\to 0} t^\alpha \frac{dx_i}{dt} = T_i \qquad \text{and} \qquad \lim_{t\to 0} x_i(t) = 0 \tag{15}$$

for $i = 1,\ldots, n$ and $(T_1,\ldots, T_n) \in W'$.

Proof. Let P be the space of differentiable paths $x(t) \in R^n$ with parameter $t \in [0, \varepsilon']$ for which $\lim_{t\to 0} x(t) = 0$, $\lim_{t\to 0} t^\alpha(dx_i/dt) = T_i \in W'$ and such that the following five conditions hold:

$$\#1 \quad \sum_{i=1}^{n-1} x_i(t)^2 \leqslant \lambda^2 x_n(t)^2, \tag{16}$$

$$\#2 \quad \left| t^\alpha \frac{dx_i}{dt}\right| \leqslant \frac{\tilde{\lambda}\delta}{n} \qquad \text{for some } \delta \leqslant 1 \text{ and all } t \in (0, \varepsilon'] \text{ for } i \neq n, \tag{17}$$

$$\#3 \quad \delta \leqslant t^\alpha \frac{dx_n}{dt} \leqslant 2\delta, \tag{18}$$

$$\#4 \quad |U_i(x)(t)| \leqslant \frac{\lambda\delta}{n}, \qquad t \in (0, \varepsilon'] \text{ and } 1 \leqslant i \leqslant n-1, \tag{19}$$

$$\#5 \quad \delta \leqslant U_n(x)(t) \leqslant 2\delta, \tag{20}$$

where $U_i(x)$ is a path for $t \in [0, \varepsilon']$ in R^n defined by

$$U_i(x)(t) \equiv \lim_{t\to 0} t^\alpha \frac{dx_i}{dt} + \int_0^t G_i\left[F_1(x),\ldots, F_N(x), s^\alpha \frac{dx_1}{ds},\ldots, s^\alpha \frac{dx_n}{ds}\right] ds. \tag{21}$$

For simplicity, let G be redefined as $\hat{G}$ so as to absorb the s^α.

Step 1: To show that $P \neq 0$, consider a "radial" path x given by $x_i \equiv t^{1-\alpha} C_i$ for $i = 1,\ldots, n$ such that $(C_1,\ldots, C_n) \in W'$. Now $|U_i(x)(t)| \leqslant |C_i| + \int_0^t |\hat{G}_i[F_1(x),\ldots, F_N(x), C_1,\ldots, C_n]|\, ds \leqslant |C_i| + \sum_{i=1}^N K_i \int_0^t |F_i(x)|\, ds$. By hypothesis and Lemma 1 the latter sum can be made arbitrarily small: $\exists$ a uniform ε' such that $\forall t < \varepsilon'$, $\sum_{i=1}^N K_i \int_0^t |F_i(x)|\, ds \leqslant ((\lambda - \tilde{\lambda})\delta)/n$. Since $|C_i| \leqslant \tilde{\lambda}\delta/n$, for such an ε, $|U_i(x)(t)| \leqslant \lambda\delta/n$ for $i = 1,\ldots, n-1$, and since $\delta \leqslant T_n \leqslant 2\delta$, we have $\delta \leqslant (1 + \lambda - \tilde{\lambda})\delta - ((\lambda - \tilde{\lambda})/n)\delta \leqslant U_n(x)(t) \leqslant$

$(2-\lambda+\bar{\lambda})\delta+((\lambda-\bar{\lambda})/n)\delta \leqslant 2\delta$. Note that at each step of the proof, ε' may have to decrease.

Step 2: Next, we show that P is a complete metric space in the metric d

$$d(x,y) \equiv \max\left\{\sup_{\substack{0\leqslant t\leqslant\varepsilon'\\1\leqslant i\leqslant n}}(|x_i(t)-y_i(t)|),\ \sup_{\substack{0<t\leqslant\varepsilon'\\1\leqslant i\leqslant n}}\left(t^\alpha\left|\frac{dx_i}{dt}-\frac{dy_i}{dt}\right|\right)\right\}.$$

The limit of any Cauchy sequence exists with respect to properties #1, #2 and #3 in the space defined by #1, #2 and #3 and the given initial conditions, and clearly any such limit also has these initial conditions. Since the $U_i(x)(t)$ for $i=1,\dots,n$ depend continuously on x in the topology induced by this metric, the subset of paths satisfying #4 and #5 in addition must be closed.

Step 3: To show that the operator V defined by $V_i(x) \equiv t^{-\alpha}\int_0^t U_i(x)(s)\,ds$ for $i=1,\dots,n$ is a contraction on P, note $\lim_{t\to 0}|t^\alpha(dV_i/dt)(x)(t)| = \lim_{t\to 0}|t^{2\alpha}(dx_i/dt)| \leqslant \bar{\lambda}\delta/n$ for $i=1,\dots,n-1$. Secondly, $|U_i(V(x))(t)| \leqslant |\lim_{t\to 0}(dV_i(x))/dt| + |\int_0^t \hat{G}_i[F_1(V(x)),\dots,F_N(V(x)),U_1(x),\dots,U_n(x)]\,ds| \leqslant \lim_{t\to 0}|t^\alpha(dx_i/dt)| + \sum_{i=1}^N\int_0^t L_i|F_i(x)| + K_i|F_i(V(x))|\,ds \leqslant (\lambda-\bar{\lambda})/n' + \bar{\lambda}\delta/n$ uniformly for some $\varepsilon''\leqslant\varepsilon'$, since $V_n(x)(t)\geqslant\delta t$. Furthermore, $\delta\leqslant(1+\lambda-\bar{\lambda})-((\lambda-\bar{\lambda})/n)\delta\leqslant U_n(V(x))(t)\leqslant(2-\lambda+\bar{\lambda})\delta+((\lambda-\bar{\lambda})/n)\delta\leqslant 2\delta$. Hence $V\colon P\to P$, possibly for smaller ε'. Let $x\in P$. $t^\alpha(dx_n/dt)\geqslant 2\delta$ implies $x_n(t)\geqslant\delta t$. Similarly, for $y\in P$, $y_n(t)\geqslant\delta t^\alpha$. Therefore the straight-line path γ_t from $x(t)$ to $y(t)$ parametrized by arc-length has the property $(\gamma_t(s))\geqslant\delta t^\beta$ with $\beta\equiv 1-\alpha$. Moreover, $|x_i(t)-y_i(t)|\leqslant\int_0^t|dx_i/ds-dy_i/ds|\,ds\leqslant\sup_{0<s\leqslant 1}|dx_i/ds-dy_i/ds|$, $i=1,\dots,n$. Thus path length

$$\begin{aligned}D &\equiv \sqrt{\sum_{i=1}^n(x_i(t)-y_i(t))^2}\\ &\leqslant t\sup_{0\leqslant t\leqslant 1}\sqrt{\sum_{i=1}^n\left(\frac{dx_i}{dt}-\frac{dy_i}{dt}\right)^2} < t^\beta n d(x,y).\end{aligned}$$

Since $\gamma_t(s)$ is parametrized by arc-length, $|d(\gamma_t(s))_j/ds|=1$; hence

$$\begin{aligned}|F_i(x(t))-F_i(y(t))| &\leqslant \int^D\left|\frac{dF_i(\gamma_t(s))}{ds}\right|ds\\ &< \max_{\gamma_t}\left\{\sum_{j=1}^n\left|\left(\frac{\partial F_i}{\partial x_j}\right)(\gamma_t(s))\frac{d(\gamma_t(s))_j}{ds}\right|\right\}\\ &< nt^\beta d(x,y)\max_{1\leqslant j\leqslant n}\left(\max_{\gamma_t}\left|\frac{\partial F_i}{\partial x_j}\right|\right).\end{aligned}$$

Finally,

$$|V_i(x)(t) - V_i(y)(t)| \leqslant \int_0^t |U_i(x)(s) - U_i(y)(s)|$$

and

$$\begin{aligned}
&|U_i(x)(t) - U_i(y)(t)| \\
&\leqslant \int_0^t \left| \hat{G}_i \left[F_1(x),..., F_N(x), \frac{dx_1}{dr},..., \frac{dx_n}{dr} \right] \right. \\
&\left. - \hat{G}_i \left[F_1(y),..., F_N(y), \frac{dy_1}{dr},..., \frac{dy_n}{dr} \right] \right| dr \\
&\leqslant \int_0^t \sum_{i=1}^N K_i |F_i(x) - F_i(y)| + \sum_{i=1}^N L_i \left| \frac{dy_i}{dt} - \frac{dx_i}{dr} \right| dr \\
&\leqslant n \left\{ \sum_{i=1}^N K_i \int_0^t t^\beta \max_{1 \leqslant j \leqslant n} \left(\max_{\delta r} \left| \frac{\partial F_i}{\partial x_j} \right| \right) + \sum_{j=1}^n L_i 2\tilde{\lambda}\delta \, dr \right\} d(x, y).
\end{aligned}$$

Again, reduce $\varepsilon' \leqslant \varepsilon''$ so that $s \leqslant \varepsilon''$ implies that the latter expression is strictly less than 1. We have thus produced a K such that $0 < K < 1$ and $d(V(x), V(y)) \leqslant Kd(x, y)$. The proof is completed by observing that the resulting unique fixed point of V is a solution of (14) with initial conditions (15).

Note. It is possible to generalize the above theorem to functionals such that

$$\begin{aligned}
&\| G_k(A_1,..., A_N, C_1,..., C_n) - G_k(B_1,..., B_N, D_1,..., D_n)\| \\
&\qquad \sum_{i=1}^N K_i(C, D) \| A_i - B_i \| + \sum_{i=1}^N L_i(A, B) \| C_i - D_i \|^{\alpha_i}, \qquad (22)
\end{aligned}$$

for $0 < \alpha_i < 1$.

COROLLARY. *Under the hypotheses of Theorem* 2, *but with the following in place of* (13).

$$\int_0^\epsilon s^{\beta-1} \max_s \left| \frac{\partial F_i}{\partial x_j} \right| ds < \infty, \qquad (23)$$

solutions of (14) *vary differentiably with respect to initial conditions* (15).

Proof. Essentially the same arguments of the proof of Theorem 2 can be applied to the equation of first variation. This is tedious and will not be included.

With $\gamma\colon [0, 1] \to M$ parametrized by t proportional to arc-length, the *energy is*

$$E_{pq}(\gamma) \equiv \int_0^1 g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) dt. \tag{24}$$

The *Hessian* is defined as follows: consider two vector fields X_1 and X_2 along a path ν; construct a parameter variation $\mu\colon U \times [0, 1] \to M$ such that

$$\mu(0, 0, t) = \nu(t), \qquad \frac{\partial \mu}{\partial s_1}(0, 0, t) = X_1(t), \qquad \frac{\partial \mu}{\partial s_2}(0, 0, t) = X_2(t), \tag{25}$$

(s_1, s_2) being a pair in the neighborhood U of the origin in R^2; and let

$$H(X_1, X_2) \equiv \left.\frac{\partial^2 E_{pq}(\mu(s_1, s_2, t))}{\partial s_1\, \partial s_2}\right|_{(0,0)} \tag{26}$$

be the Hessian of E_{pq} at γ. Points p and q are said to be *conjugate* along a geodesic γ if there exists a vector field X along γ vanishing at p and q which is parallel with respect to the Riemannian connection, in other words, which has covariant derivative 0 with respect to the tangent vector field of the geodesic (i.e., satisfies Eq. (3) in appropriate coordinates). X is called a *Jacobi field.* The null space of the Hessian consists precisely of all Jacobi fields along γ. Every Jacobi field along γ is the restriction to γ of a variation vector field whose variation paths near γ are geodesics. The dimension of the null space of H is the *multiplicity* of p and q as conjugate points. The *index* of H is the maximum dimension of a subspace of the vector fields along γ on which H is negative definite. The Morse Index Theorem is extended to topological manifolds which are Riemannian except at a collection of isolated points at which some of the Christoffel symbols have infinite singularities in any coordinates, but at which condition (23) holds.

THEOREM 3. *If $\gamma\colon [0, 1] \to M$ is a path which is a geodesic on M with $\gamma(0) = p$ and $\gamma(1) = q$ removed, and if for some ε and ε'*

$$\int_0^{\epsilon} \max_s |\Gamma^i_{jk}| < \infty \qquad \text{and} \qquad \int_{\epsilon'}^1 \max_s |\Gamma^i_{jk}| < \infty, \tag{27}$$

then the index of H for γ is finite.

COROLLARY. *On a differentiable manifold M with finitely many isolated point singularities P_α of the Riemannian connection satisfying* (23), *the Morse Index is finite for a path which is piecewise-geodesic on $M - \{P_\alpha\}$.*

COROLLARY. *Given a generalized instability such that* (1) *incoming trajectories have limit tangent vectors of outgoing trajectories at all stationary unstable points, and* (2) *the geodesic associated system has property* (23) *then there can be only a finite number of unstable manifolds of dimension* >1 *along any trajectory between two points.*

In the case of algebraic varieties, the limit tangent vector can be regarded as a vector in R^n at 0 which is the limit of the tangent vectors to γ as a singularity 0 is approached.

In the following theorem it will be necessary to know that the Zariski tangent cone at 0 "really is tangent to the variety" at 0.

LEMMA 3. *If any two-dimensional plane* Π *intersects the cone* Z *in one of its generating lines* L *through* 0, *then there is a branch,* M, *of* $V \cap \Pi$ *which is tangent to* L *at* 0 *(*M *must be one-dimensional). Moreover, if coordinates* $(z_1,..., z_n)$ *are chosen so that the* z_n*-axis is contained in* Π, *and* L *is given parametrically by* $z_i = C_i r$, $i = 1,..., n-1$, $z_n = C_n r + D_n r^\alpha$ *for some* $\alpha > 1$. *The constants* C_n *and* D_n *are continuous functions of the position of* Π *and* α *is constant for small rotations of* Π *about the* z_n*-axis.*

Proof. Suppose V is the set of zeros of the multi-series f, and let F be the lowest-order form of f. The two-dimensional plane Π can be given by the system of equations $a_{ij}x_i = 0$, where $j = 1,..., n-2$ and $i = 1,..., n$. If S_r is the sphere of radius r centered at 0 and $P_r: S_r \to S_1$ is the projection taking the point $(x_1,..., x_n)$ on S_r to the point $(x_1,..., x_n)$ on S_1, then $P_r(S_r \cap V) = \{(x_1/r,..., x_n/r): \sum x_i^2 = r^2 \text{ and } f(x_1,..., x_n) = 0\} = \{(x_1,..., x_n): \sum x_i^2 = 1 \text{ and } f(rx_1,..., rx_n) = 0\}$. Now the set of zeros of $f(rx_1,..., rx_n)$ is the same as the set of zeros of $f(rx_1,..., rx_n)/r^m$, where m is the total degree of F. Since $f(rx_1,..., rx_n)/r^m = F(x_1,..., x_n) + r$ (higher-order terms) approaches $F(x_1,..., x_n)$ coefficient-wise as $r \to 0$, $P_r(S_r \cap V)$ approaches $S \cap Z$ point-wise. Since $\Pi \cap Z$ is a line through the origin, say L, $\Pi \cap S \cap Z \neq 0$. So for some small ε and all $r < \varepsilon$, $\Pi \cap P_r(S_r \cap V) \neq 0$ and approaches $\Pi \cap S \cap Z$ point-wise. Since $\Pi \cap V$ can have only finitely many components near 0, and since $\Pi \cap V$ has points arbitrarily close to 0, it must be one-dimensional near 0. Since $P_r(S_r \cap V) \to S \cap Z$ point-wise, one of the branches, say C, of $\Pi \cap V$ must be tangent to L. Let $z_i = \Theta_i(\theta_1,..., \theta_{n-2})r$, $i = 1,..., n-1$, be polar coordinates in the $(x_1,..., x_{n-1})$-plane. Since Π contains the z_n-axis, this tangency can be expressed as $z_i = C_i r$, $i = 1,..., n-1$, $z_n = C_n r + D_n r^\alpha$ for some $\alpha > 1$, a classical result about tangents to analytic curves, where α depends only on the lowest order in the multi-series which has $\Pi \cap V$ as zeros in the plane Π. This order remains constant for $\Pi \cap V$ and C_n and D_n are continuous functions $\theta = (\theta_1,..., \theta_n)$ for θ in some small neighborhood of θ_0 such that $\Pi_{\theta_0} = \Pi_\theta$. Q.E.D.

In the following theorem, a piece of Z will be called "wedge-shaped" if it has the form $\{(rz_1,..., rz_n): 0 < r < 1$ and $(z_1,..., z_n) \in C\}$, where C is the interior of an $n-2$ cell contained in $Z \cap S_\delta$, S_δ being the $n-1$ sphere of radius δ with center at 0, for some δ. "Projection in some direction" will be taken to mean a 1–1 map of one $n-1$ dimensional variety A onto another, B, which takes a point p of A to the (assumed) unique point q of B such that both p and q lie on a translation of a given fixed line L, which is the "direction."

THEOREM 4. *If there is a projection in some direction of a subset V' of V onto a wedge-shaped piece W of Z, then there exists an $n-2$ parameter, C^1 family of nonintersecting geodesics on $V' - \{0\}$ whose limit tangent vectors at 0 take all possible directions in a smaller wedge $W' \subset W$. Moreover, the geodesics vary differentiably with respect to their limit tangent vectors.*

Proof. We show that the Christoffel symbols satisfy the hypothesis of Theorem 2. Let $z_1,..., z_n$ be coordinates as above with the Euclidean inner product as metric, such that the projection mentioned in the hypothesis is in the direction of the z_n-axis, and such that 0 is at the origin. Since W is a developable surface contained in Z, it can be flattened out to a subset of R^{n-1}. As such, W can be given coordinates $x_1,..., x_{n-1}$ which are everywhere orthonormal with respect to the metric inherited by W in its role as a subvariety of R^n, such that $x_1 \to 0,..., x_{n-1} \to 0$ as the apex of the cone Z is approached on W. Consider the coordinates $z_1(x_1,..., x_{n-1}),..., z_n(x_1,..., x_{n-1})$ on V' induced by the projection in the direction of the z_n-axis. At all points $(z_1,..., z_n)$ for which the projection in the z_n-direction is defined, the implicit function theorem guarantees a function $h(z_1,..., z_{n-1})$ such that $f(z_1,..., z_{n-1}, h(z_1,..., z_{n-1})) = 0$. The metric g on V' induced by the Euclidean inner product $\cdot$ on E^n has components $(G_{/x_i} \equiv \partial G/\partial x_i)\, g_{ij} = (z_{1/x_i},..., z_{n-1/x_i}, h_{/z_k} z_{k/x_i}) \cdot (z_{1/x_j},..., z_{n-1/x_j}, h_{/z_m} z_{m/x_j}) = (z_{p/x_i})(z_{p/x_j}) + (h_{/z_k} z_{k/x_i})(h_{/z_m} z_{m/x_j})$, where the Einstein summation convention is in force over the available indices $1,..., n-1$ (whenever a term contains a pair of similarly denoted indices, it is to be regarded as a sum over that index). Similarly, the metric G on W induced by the Euclidean inner product on E^n has components $G_{ij} = (z_{1/x_i},..., z_{n-1/x_i}, H_{/z_k} z_{k/x_i}) \cdot (z_{1/x_j},..., z_{n-1/x_j}, H_{/z_k} z_{k/x_j}) = (z_{p/x_i})(z_{p/x_j}) + (H_{/z_k} z_{k/x_i})(H_{/z_m} z_{m/x_j})$, where $H(z_1,..., z_n)$ is the function such that $F(z_1,..., z_{n-1}, H(z_1,..., z_{n-1})) = 0$. Applying the product rule for differentiation over and over again, one obtains Γ^i_{jk/x_p} on V' in terms of coordinates $x_1,..., x_{n-1}$. Noting that G_{ij} has exactly the same expression as that of g_{ij} except that H is replaced by h everywhere it occurs, one can then obtain the expression for Γ^i_{jk/x_p} on W by substituting H for h everywhere it occurs in Γ^i_{jk/x_p} on V'. The strategy of the proof is to show that $s \cdot \max_s |\Gamma^i_{jk/x_p}(\text{on } V') - \Gamma^i_{jk/x_p}(\text{on } W)|$ is integrable from 0 to some small

TABLE I

Decomposition of $|\Gamma^i_{jk/x_p}(\text{on } V') - \Gamma^i_{jk/x_p}(\text{on } W)|$ via the Triangle Inequality into 19 Differences Which are $\leqslant o(\alpha - 2)$

$$\left|\frac{\Pi^{n-2}g}{\det(g)} - \frac{\Pi^{n-2}G}{\det(G)}\right| \begin{bmatrix} z_{i/x_j/x_m/x_p} z_{i/x_k} \\ \text{or} \\ h_{z_m z_p} h_{z_q} z_{m/x_t} z_{p/x_s} z_{q/x_u} \\ \text{or} \\ h_{z_m} h_{z_q} z_{m/x_j/x_p} z_{q/x_k} \end{bmatrix}^2 |\det(z_{u/x_v})^{-2}$$

$$\left|\frac{\Pi^{2(n-1)}g}{\det(g)^2} - \frac{\Pi^{2(n-1)}G}{\det(G)^2}\right| \left[\frac{1}{\det(g_{ij})}\right]_{/x_m} \det(z_{u/x_v})^{-4}$$

$$\left|\frac{\Pi^{n-1}g}{\det(g)} - \frac{\Pi^{n-1}G}{\det(G)}\right| \begin{bmatrix} z_{i/x_j/x_p/x_m} z_{i/x_j/x_k} \\ \text{or} \\ z_{i/x_j/x_p} z_{i/x_k/x_m} \\ \text{or} \\ h_{z_m z_p z_s} h_{z_q} z_{m/x_j} z_{p/x_k} z_{s/x_t} z_{q/x_u} \\ \text{or} \\ h_{z_m z_p} h_{z_q} z_{m/x_j/x_p/x_s} z_{q/x_k} \\ \text{or} \\ h_{z_m z_p} h_{z_q z_s} z_{m/x_j} z_{p/x_k} z_{q/x_t} z_{s/x_u} \\ \text{or} \\ h_{z_m} h_{z_q} z_{m/x_j/x_p} z_{q/x_k/x_s} \end{bmatrix} \det(z_{u/x_v})^{-2}$$

[a] One term is to be chosen from a bracketed factor, and two terms from a squared factor.

positive ε, and then, since W is flat, $\Gamma^i_{jk/x_p} = 0$ on W, so $s \cdot \max_s \Gamma^i_{jk/x_p}(\text{on } V')$ is also integrable from 0 to ε. The rationale for looking at this difference is that, although $\Gamma^i_{jk/x_p}(\text{on } V')$ is somewhat complicated, one can break up the difference via the triangle inequality into a sum composed of 19 types of differences. Using the fact that $\det(z_{u/x_v})$, which occurs in the denominators of these differences is bounded, a case-by-case analysis will show that the differences are $o(\alpha - 2)$. α is greater than 1, so $\max_s |\text{differences}|$ is $o(\beta)$ for $\beta > -1$, hence is Lebesgue-integrable from 0 to ε.

Note. Theorem 4 can be generalized to analytic varieties of other codimensions by performing several projections of the variety in succession onto tangent plans of increasingly smaller dimension (one less each time) until the Zariski tangent cone is reached. (We are indebted to A. Wallace for suggesting that this might be so.)

Concluding Remarks

Some of theory presented has found application to various problems. Theorem 2 can used in the construction of a generalized exponential map at singularities where (13) or (23) hold, and then by Lemma 1 it is seen that a wide range of geometrical field equations provide integrable Christoffel symbols at isolated intrinsic singularities. With the exponential, an invariance group can be constructed in a neighborhood of such a singularity. It is hoped that such an approach will lead to a classification of particles represented as Riemannian singularities, which are structurally stable.

Finally, in unpublished work, we have shown that the results on uniqueness and differentiability of trajectories with respect to initial conditions on the limit tangent vectors carry over to the analytic case on analytic varieties. If outgoing trajectories are given to vary analytically w.r.t. limit tangent vectors at stationary unstable points (in any wedge-shaped set), then Christoffel symbols of the associated geodesic system obey (23). We have also shown that the integral Riemannian curvature over a relatively compact neighborhood containing an isolated singularity of an analytic variety is finite. This is done by showing that the neighborhood can be covered by a finite set of wedge-shaped sets satisfying the conditions of Theorem 4. Various other isolated results have been obtained, including a generalized Gauss–Bonnet formula. Thus, the geometric approach advocated here appears to be fruitful in a number of applications.

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